Johns Hopkins Math Tournament 2021

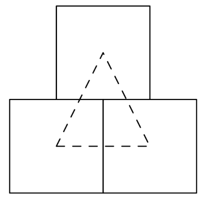
Individual Round: Geometry

April 3rd, 2021

Instructions

- **Remember you must be proctored while taking the exam.**
- This test contains 10 questions to be solved individually in 60 minutes.
- All answers will be integers.
- Problems are weighted relative to their difficulty, determined by the number of students who solve each problem.
- No outside help is allowed. This includes people, the internet, translators, books, notes, calculators, or any other computational aid. Similarly, graph paper, rulers, protractors, compasses, and other drawing aids are not permitted.
- If you believe the test contains an error, immediately tell your proctor.
- Good luck!

1. In the diagram below, a triangular array of three congruent squares is configured such that the top row has one square and the bottom row has two squares. The top square lies on the two squares immediately below it. Suppose that the area of the triangle whose vertices are the centers of the three squares is 100. Find the area of one of the squares.



2. A triangle is *nondegenerate* if its three vertices are not collinear. A particular nondegenerate triangle $\triangle JHU$ has side lengths x, y, and z, and its angle measures, in degrees, are all integers. If there exists a nondegenerate triangle with side lengths x^2, y^2, and z^2, then what is the largest possible angle measure in the original triangle $\triangle JHU$, in degrees?

3. Let $ABCDEF$ be a convex hexagon such that $AB = CD = EF = 20$, $BC = DE = FA = 21$, and $\angle A = \angle C = \angle E = 90^\circ$. The area of $ABCDEF$ can then be expressed in the form $a + \frac{b\sqrt{c}}{d}$, where a, b, c, and d are positive integers, b and d are relatively prime, and c is not divisible by the square of any prime. Find $a + b + c + d$.

4. Triangle ABC has side lengths $AC = 3$, $BC = 4$, and $AB = 5$. Let R be a point on the incircle ω of $\triangle ABC$. The altitude from C to $\overline{AB}$ intersects ω at points P and Q. Then, the greatest possible area of $\triangle PQR$ is $\frac{m\sqrt{n}}{p}$, where m and p are relatively prime positive integers, and n is a positive integer not divisible by the square of any prime. Find $m + n + p$.

5. Let $\mathcal{S}$ be the set of points (x, y) in the Cartesian coordinate plane such that $xy > 0$ and $x^2+y^2+2x+4y \leq 2021$. The total area of $\mathcal{S}$ can be expressed in simplest form as $a\pi + b$, where a and b are integers. Compute $a + b$.

6. $JHMT$ is a convex quadrilateral with perimeter 68 and satisfies $\angle HJT = 120^\circ$, $HM = 20$, and $JH + JT = JM > HM$. Furthermore, ray $\overrightarrow{JM}$ bisects $\angle HJT$. Compute the length of $\overline{JM}$.

7. Triangle $\triangle JHT$ has side lengths $JH = 14$, $HT = 10$, and $TJ = 16$. Points I and U lie on $\overline{JH}$ and $\overline{JT}$, respectively, so that $HI = TU = 1$. Let M and N be the midpoints of $\overline{HT}$ and $\overline{IU}$, respectively. Line $\overleftrightarrow{MN}$ intersects another side of $\triangle JHT$ at a point P other than M. Compute MP^2.

8. Triangle $\triangle ABC$, with $BC = 48$, is inscribed in a circle Ω of radius $49\sqrt{3}$. There is a unique circle ω that is tangent to $\overline{AB}$ and $\overline{AC}$ and internally tangent to Ω. Let D, E, and F be the points at which ω is tangent to Ω, $\overline{AB}$, and $\overline{AC}$, respectively. The rays $\overrightarrow{DE}$ and $\overrightarrow{DF}$ intersect Ω at points X and Y, respectively, such that $X \neq D$ and $Y \neq D$. Compute the length of $\overline{XY}$.

9. Right triangle $\triangle ABC$ has a right angle at A. Points D and E respectively lie on $\overline{AC}$ and $\overline{BC}$ so that $\angle BDA \cong \angle CDE$. If the lengths DE, DA, DC, and DB in this order form an arithmetic sequence of distinct positive integers, then the set of all possible areas of $\triangle ABC$ is a subset of the positive integers. Compute the smallest element in this set that is greater than 1000.

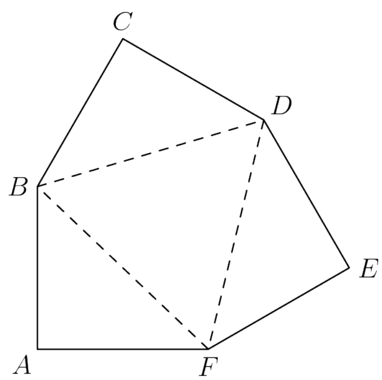
10. Parallelogram $JHMT$ satisfies $JH = 11$ and $HM = 6$, and point P lies on $\overline{MT}$ such that JP is an altitude of $JHMT$. The circumcircles of $\triangle HMP$ and $\triangle JMT$ intersect at the point $Q \neq M$. Let A be the point lying on $\overline{JH}$ and the circumcircle of $\triangle JMT$. If $MQ = 10$, then the perimeter of $\triangle JAM$ can be expressed in the form $\sqrt{a} + \frac{b}{c}$, where a, b, and c are positive integers, a is not divisible by the square of any prime, and b and c are relatively prime. Find $a + b + c$.

Geometry Solutions

1. The length of the horizontal base and the vertical height of the triangle both equal the side length of each square. Therefore, if we let this side length be s, then the area of the triangle is $\frac{1}{2}s^2 = 100$. The area of each square is therefore $s^2 = \boxed{200}$.

2. The triangle inequality states that three line segments of positive lengths x, y, and z form a nondegenerate triangle if and only if $x+y > z$, $y+z > x$, and $z+x > y$. Therefore, in order for a nondegenerate triangle to have side lengths x^2, y^2, and z^2, we need $x^2 + y^2 > z^2$, $y^2 + z^2 > x^2$, and $z^2 + x^2 > y^2$. By the Pythagorean Theorem, these three inequalities tell us that no one side (of length x, y, or z) is long enough to be a hypotenuse of a right triangle with the other two sides as legs. This means that the nondegenerate triangle with side lengths x, y, and z must be acute. Conversely, knowing that the x-y-z triangle is acute is enough to guarantee that the x^2-y^2-z^2 triangle is nondegenerate. Hence, the largest possible integer angle measure of the original acute triangle is $\boxed{89}$ degrees.

3. **Solution:**



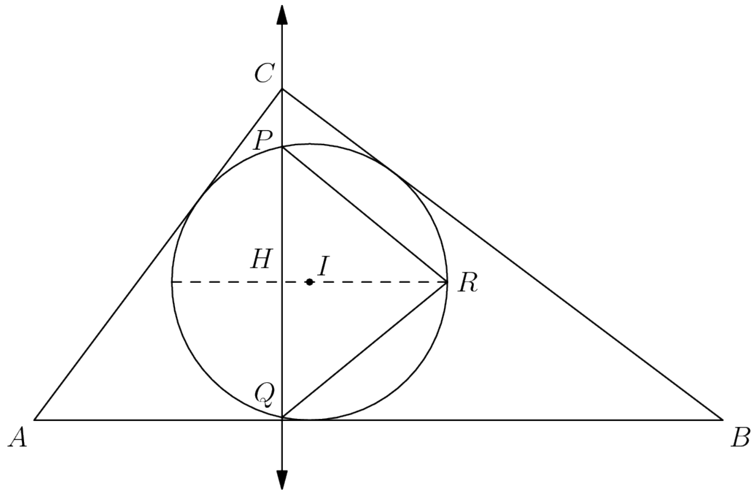
Observe that upon drawing $\overline{BD}$, $\overline{DF}$, and $\overline{FB}$, $ABCDEF$ is simply an equilateral triangle with sides of length $\sqrt{20^2 + 21^2} = 29$ along with three right triangles with legs of length 20 and 21. Therefore, the area of $ABCDEF$ is

$$\frac{\sqrt{3}}{4}(29^2) + \frac{3}{2}(20)(21) = 630 + \frac{841\sqrt{3}}{4},$$

where we have used the formula for the area of an equilateral triangle. The requested sum is $630 + 841 + 3 + 4 = \boxed{1478}$.

4. Triangle ABC has side lengths $AC = 3$, $BC = 4$, and $AB = 5$. Let R be a point on the incircle ω of $\triangle ABC$. The altitude from C to $\overline{AB}$ intersects ω at points P and Q. Then, the greatest possible area of $\triangle PQR$ is $\frac{m\sqrt{n}}{p}$, where m and p are relatively prime positive integers, and n is a positive integer not divisible by the square of any prime. Find $m + n + p$.

   Solution:



Let I be the incenter of $\triangle ABC$. Let $\overline{PQ}$ and $\overleftrightarrow{RI}$ intersect at H. It suffices to maximize RH, the height of $\triangle PQR$.

Note that R must be on the right side of $\overleftrightarrow{PQ}$ because $\angle ACP = \cos^{-1}\left(\frac{4}{5}\right) < 45^\circ = \angle ACI$, so more than half of ω will be to the right of $\overleftrightarrow{PQ}$, so we may generate a greater value for RH.

Furthermore, R, I, and H must be collinear, as the longest chord in a circle is its diameter, so the chord determined by these 3 points will give the optimal RH.

Let r be the inradius of $\triangle ABC$, and let a, b, and c be the side lengths of the triangle. The area of $\triangle ABC$ is $r\left(\frac{a+b+c}{2}\right) = 6r = 6$, so $r = 1$. Because $\triangle CHI$ is right,

$$HI^2 = CI^2 - CH^2 = 2 - \left(\frac{12}{5} - 1\right)^2 = \frac{1}{25} \implies HI = \frac{1}{5}.$$

Because $\triangle PHI$ is right,

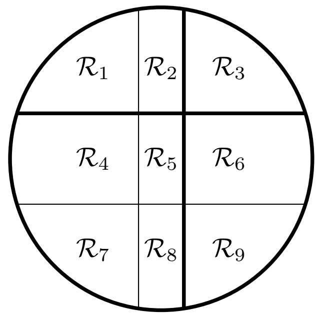
$$PH^2 = PI^2 - HI^2 = 1 - \frac{1}{25} = \frac{24}{25} \implies PH = \frac{2\sqrt{6}}{5}.$$

By symmetry, $PH = HQ$, so $PQ = \frac{4\sqrt{6}}{5}$. From our earlier computation, $HR = HI + IR = \frac{1}{5} + 1 = \frac{6}{5}$. Therefore, the greatest possible area of $\triangle PQR$ is

$$\frac{1}{2}(PQ)(HR) = \frac{1}{2}\left(\frac{4\sqrt{6}}{5}\right)\left(\frac{6}{5}\right) = \frac{12\sqrt{6}}{25}.$$

The requested sum is $12 + 6 + 25 = \boxed{43}$

5. The inequality $x^2 + y^2 + 2x + 4y \leq 2021$ is equivalent to $(x+1)^2 + (y+2)^2 \leq 2026$, which describes a circular disk of radius $\sqrt{2026}$ centered at $(-1, -2)$. Call this disk $\mathcal{D}$. The four lines specified by $x = 0$, $x = -2$, $y = 0$, and $y = -4$ partition $\mathcal{D}$ into nine smaller regions, as shown in the diagram below (which is not drawn to scale, though scale is not important for this problem). The boundary of the disk and the lines given by $x = 0$ and $y = 0$ are boldfaced.



The set of points (x, y) in $\mathcal{D}$ with $xy > 0$ can be expressed as the union of five of these regions (excluding certain points on the boundary): $\mathcal{R}_3$, $\mathcal{R}_4$, $\mathcal{R}_5$, $\mathcal{R}_7$, and $\mathcal{R}_8$. Use the notation $[\mathcal{R}]$ to denote the area of a two-dimensional region $\mathcal{R}$. Because of the congruence relationships $\mathcal{R}_1 \cong \mathcal{R}_3 \cong \mathcal{R}_7 \cong \mathcal{R}_9$, $\mathcal{R}_2 \cong \mathcal{R}_8$, and $\mathcal{R}_4 \cong \mathcal{R}_6$, we have

$$[\mathcal{D}] = \sum_{k=1}^{9}[\mathcal{R}_k] = [\mathcal{R}_5] + 2([\mathcal{R}_3] + [\mathcal{R}_4] + [\mathcal{R}_7] + \mathcal{R}_8]) = 2([\mathcal{R}_3] + [\mathcal{R}_4] + [\mathcal{R}_5] + [\mathcal{R}_7] + \mathcal{R}_8]) - [\mathcal{R}_5].$$

Therefore, the area of the desired set of points is

$$[\mathcal{R}_3 \cup \mathcal{R}_4 \cup \mathcal{R}_5 \cup \mathcal{R}_7 \cup \mathcal{R}_8] = [\mathcal{R}_3] + [\mathcal{R}_4] + [\mathcal{R}_5] + [\mathcal{R}_7] + [\mathcal{R}_8] = \frac{[\mathcal{D}] + [\mathcal{R}_5]}{2}.$$

For the current problem, $\mathcal{D}$ has radius $\sqrt{2026}$, so $[\mathcal{D}] = 2026\pi$, and $\mathcal{R}_5$ is a 2-by-4 rectangle, so $[\mathcal{R}_5] = 8$. Hence, $\frac{[\mathcal{D}]+[\mathcal{R}_5]}{2} = 1013\pi + 4$, so the answer is $\boxed{1017}$.

6. Because $JM > HM > JH$, there exists a unique point A on $\overline{JM}$ such that $\angle JHA = 60^\circ$. Triangle $\triangle JHA$ is equilateral, so $AJ = AH = JH$, so $AM = JM - AJ = JM - JH = JT$. We also have $\angle HAM = \angle HJT = 120^\circ$, so $\triangle HAM \cong \triangle HJT$. This congruence and the fact $\angle JHA = 60^\circ$ imply $\angle THM = 60^\circ$. Therefore, $\triangle THM$ is equilateral, so $TM = HM = 20$. Since the perimeter of $JHMT$ is 68,
$$68 = (JH + JT) + HM + TM = JM + 20 + 20,$$
so $JM = 68 - 20 - 20 = \boxed{28}$.

7. For real numbers $t \in [0, 14]$, let I_t and U_t be the points on $\overline{JH}$ and $\overline{JT}$, respectively, satisfying $HI_t = TU_t = t$, and let N_t be the midpoint of $\overline{I_tU_t}$ (note that $N_0 = M$ and $N_1 = N$). We claim that $N_{14} = P$. First, because $I_{14} = J$, both I_{14} and U_{14} lie on $\overline{JT}$, so N_{14} lies on $\overline{JT}$. It remains to show that N_{14}, N_1, and N_0 are all collinear. More generally, we will show that any three distinct points of the form N_x, N_y, and N_z are all collinear.

   Without loss of generality, let $0 \le x < y < z \le 14$. Treating the points on the plane as two-dimensional vectors, we have $I_y = \frac{z-y}{z-x}I_x + \frac{y-x}{z-x}I_z$ and $U_y = \frac{z-y}{z-x}U_x + \frac{y-x}{z-x}U_z$. For any $t \in [0, 14]$, we have $N_t = \frac{1}{2}I_t + \frac{1}{2}U_t$, so

$$\begin{aligned} N_y = \frac{1}{2}I_y + \frac{1}{2}U_y &= \frac{1}{2}\left(\frac{z-y}{z-x}I_x + \frac{y-x}{z-x}I_z\right) + \frac{1}{2}\left(\frac{z-y}{z-x}U_x + \frac{y-x}{z-x}U_z\right) \\ &= \frac{z-y}{z-x}\left(\frac{1}{2}I_x + \frac{1}{2}U_x\right) + \frac{y-x}{z-x}\left(\frac{1}{2}I_z + \frac{1}{2}U_z\right) = \frac{z-y}{z-x}N_x + \frac{y-x}{z-x}N_z. \end{aligned}$$

   Since N_y is a weighted average of N_x and N_z, N_y lies on $\overline{N_xN_z}$.

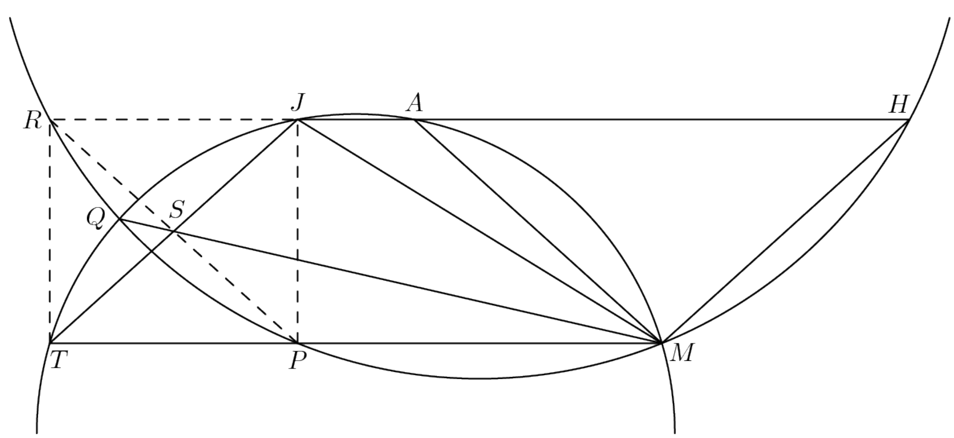
   This means that $M = N_0$, $N = N_1$, and N_{14} are collinear, so N_{14} must equal P. Thus, P is the midpoint of $\overline{I_{14}U_{14}}$ and satisfies $TP = TU_{14} + U_{14}P = 14 + \frac{16-14}{2} = 15$. By the Law of Cosines, we have
$$MP^2 = TM^2 + TP^2 - 2TM \cdot TP \cdot \cos\angle T.$$

We also know
$$JH^2 = TH^2 + TJ^2 - 2TH \cdot TJ \cdot \cos\angle T.$$
Plugging in $JH = 14$, $TJ = 16$, $TH = 10$, $TM = 5$, and $TP = 15$ yields $\cos\angle T = \frac{1}{2}$ and
$$MP^2 = 25 + 225 - 2 \cdot 5 \cdot 15 \cdot \frac{1}{2} = 250 - 75 = \boxed{175}.$$

8. Let $\theta = \angle BAC$. Let P and Q be the centers of Ω and ω, respectively. Then, $\angle BPC = 2\theta$, $\angle EQF = 180^\circ - \angle EAF = 180^\circ - \angle BAC = 180^\circ - \theta$, and $\angle XDY = \angle EDF = \frac{1}{2}\angle EQF = 90^\circ - \frac{\theta}{2}$. With r denoting the radius of Ω, we have the formula $XY = 2r\sin\angle XDY$, so
$$XY = 2r\sin\left(90^\circ - \frac{\theta}{2}\right) = 2r\cos\left(\frac{\theta}{2}\right) = 2r\sqrt{\frac{1+\cos\theta}{2}}.$$
We also know $BC = 2r\sin\angle BAC = 2r\sin\theta$. Thus, $\sin\theta = \frac{BC}{2r}$, so $\cos\theta = \sqrt{1 - \left(\frac{BC}{2r}\right)^2}$. We now plug in $r = 49\sqrt{3}$ and $BC = 48$ and obtain
$$\cos\theta = \sqrt{1 - \left(\frac{24}{49\sqrt{3}}\right)^2} = \sqrt{\frac{49^2 - 24 \cdot 8}{49^2}} = \sqrt{\frac{2209}{49^2}} = \frac{47}{49},$$
so $XY = 2 \cdot 49\sqrt{3} \cdot \sqrt{\frac{1+47/49}{2}} = 98\sqrt{3} \cdot \sqrt{\frac{48}{49}} = \frac{98\sqrt{3}\cdot 4\sqrt{3}}{7} = 14 \cdot 4 \cdot 3 = \boxed{168}$.

9. Reflect points B and E over line $\overleftrightarrow{AC}$, and let their images be B' and E', respectively. Notice that $\triangle CBB'$ is isosceles and that B, D, and E' are collinear. To model the arithmetic sequence of lengths from the problem statement, set $DE' = DE = x$, $DA = x + d$, $DC = x + 2d$, and $DB = x + 3d$, for positive integers x and d (d must be positive because $\triangle DAB$ has hypotenuse length $DB = x + 3d$, which must exceed the leg length $DA = x + d$). We employ mass points, and in general, we will use m_P to denote the mass of some point P. First analyzing line segment $\overline{BE'}$, we assign $m_B = x$ and $m_{E'} = x+3d$ so that D is the center of mass of B and E'. Because the cevian $\overline{CA}$ is a median of $\triangle CBB'$, we set $m_B = m_{B'} = x$ and hence $m_A = m_B + m_{B'} = 2x$. Then, $m_C = m_{E'} - m_{B'} = (x+3d) - x = 3d$. The values of x and d must satisfy the center-of-mass equation $m_A \cdot DA = m_C \cdot DC$:
$$2x(x+d) = 3d(x+2d) \implies 2x^2 + 2dx = 3dx + 6d^2 \implies 2x^2 - dx - 6d^2 = 0 \implies (x-2d)(2x+3d) = 0.$$
So $x = 2d$, meaning $DE = DE' = 2d$, $DA = 3d$, $DC = 4d$, and $DB = 5d$. Applying the Pythagorean Theorem to $\triangle DAB$, we obtain $AB = 4d$. Because $AC = AD + DC = 3d + 4d = 7d$, the area of $\triangle ABC$ is $\frac{1}{2} \cdot 4d \cdot 7d = 14d^2$. The smallest integer greater than 1000 that can be written in the form $14d^2$ for an integer d is $\boxed{1134}$, attained when $d = 9$.

10. Here is a diagram for the problem:



We start by computing AM. Note that $JAMT$ is a cyclic trapezoid and hence is an isosceles trapezoid. Thus, $JT = AM = 6$.

We proceed by finding JM. Let us construct rectangle $JPTR$ (which has diagonal $\overline{JT}$), where R is a point on line JH. Then, we have that $HM = JT = PR = 6$, so $HMPR$ is an isosceles trapezoid and hence is cyclic; thus, R lies on the circumcircle of $\triangle HMP$. From here, define S to be the intersection of $JPTR$'s diagonals. Note that

$$PS \cdot SR = JS \cdot ST = 3 \cdot 3 = 9.$$

Since $\overline{PR}$ is a chord of the circumcircle of $\triangle HMP$ and $\overline{JT}$ is a chord of the circumcircle of $\triangle JMT$, we can conclude that S has equal power with respect to both of these circles, which means that S lies on the radical axis of the circles. Thus, S lies on $\overline{MQ}$. By the Power of a Point Theorem, we have

$$JS \cdot ST = MS \cdot SQ = 9.$$

Furthermore,

$$MS + SQ = 10,$$

so $\{MS, SQ\} = \{1, 9\}$. Using Stewart's Theorem on $\triangle JMT$, we have

$$MT^2 \cdot JS + JM^2 \cdot ST = MS^2 \cdot JT + JS \cdot JT \cdot ST \implies JM = \sqrt{2MS^2 - 103}.$$

If $MS = 1$, then JM is nonreal, so we must have $MS = 9$, which yields $JM = \sqrt{59}$.

We conclude by determining AJ. To do this, we employ Ptolemy's Theorem on isosceles trapezoid $JAMT$:

$$AJ \cdot MT + JT \cdot AM = JM \cdot AT \implies 11AJ + 36 = 59 \implies AJ = \frac{23}{11}.$$

Therefore, the perimeter of $\triangle JAM$ is

$$AM + JM + AJ = 6 + \sqrt{59} + \frac{23}{11} = \sqrt{59} + \frac{89}{11},$$

so the requested sum is $59 + 89 + 11 = \boxed{159}$.